A geometric approach to certain systems of exponential equations

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Three conjectures

Conjecture (Schanuel's conjecture)

If $z_1, \ldots, z_n \in \mathbb{C}$ *are* $\mathbb{Q}$*-linearly independent, then*

$$\operatorname{td}_{\mathbb{Q}} \mathbb{Q}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) \geq n$$

where td *stands for transcendence degree.*

Conjecture (Zilber's EAC conjecture)

Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ *be an irreducible* ***free*** *and* ***rotund*** *variety. Then there is a point* $\boldsymbol{z} \in \mathbb{C}^n$ *such that* $(\boldsymbol{z}, e^{\boldsymbol{z}}) \in V$.

Conjecture (Zilber's quasiminimality conjecture)

$\mathbb{C}_{\exp} := (\mathbb{C}; +, \cdot, \exp)$ *is quasiminimal, i.e. every definable subset is countable or co-countable.*

Schanuel's conjecture

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where td *stands for transcendence degree.*

- This captures the transcendence properties of exp.
- It is out of reach. For example, it implies the algebraic independence of e and π which is a long-standing open problem. To see this, set $n = 2,\ z_1 = i\pi,\ z_2 = 1$.

Exponential equations

- Which systems of equations have solutions in $\mathbb{C}_{\exp} := (\mathbb{C}; +, \cdot, \exp)$?
- Consider the system $z_1 = 2z_2 + 1, e^{z_1} = 3(e^{z_2})^2$.
- exp is a homomorphism from the additive group $(\mathbb{C}; +, 0)$ to the multiplicative group $(\mathbb{C}^\times; \cdot, 1)$, i.e. $e^{x+y} = e^x \cdot e^y$. Therefore the above system does not have a solution.
- Let $p(X, Y) \in \mathbb{Q}[X, Y]$ be a non-zero polynomial. Does the system $e^z = 1, p(e, z) = 0$ have a solution?
- This depends on the aforementioned problem on algebraic independence of e and π. If they are independent, then the above system cannot have a solution.

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- Let $p(X, Y) \in \mathbb{Q}[X, Y]$ be a non-zero polynomial. Does the system $e^z = 1, p(e, z) = 0$ have a solution?
- This depends on the aforementioned problem on algebraic independence of e and π. If they are independent, then the above system cannot have a solution.
- We can get rid of iterated exponentials. For instance, given the equation $e^{e^z} = z$, we introduce new variables z_1, z_2, w_1, w_2 and consider the system $w_1 = e^{z_1}, w_2 = e^{z_2}, w_1 = z_2, w_2 = z_1$.
- This system has a solution if and only if the original equation does.
- The system has a solution iff the variety $V \subseteq \mathbb{C}^2 \times (\mathbb{C}^\times)^2$ defined by $w_1 = z_2, w_2 = z_1$ contains an exponential point, i.e. a point $(z_1, z_2, e^{z_1}, e^{z_2})$.
- Thus, the question is: which varieties $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ intersect the graph $\Gamma := \{(\boldsymbol{z}, \exp(\boldsymbol{z})) : \boldsymbol{z} \in \mathbb{C}^n\} \subseteq \mathbb{C}^{2n}$?

Exponential Algebraic Closedness

Conjecture (EAC)

Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be an irreducible ***free*** *and* ***rotund*** *variety. Then there is a point $\boldsymbol{z} \in \mathbb{C}^n$ such that $(\boldsymbol{z}, e^{\boldsymbol{z}}) \in V$, where $\boldsymbol{z} := (z_1, \ldots, z_n)$.*

Here and later boldface letters denote tuples.

- Let Z and W be the projections of V to $\mathbb{C}^n$ and $(\mathbb{C}^\times)^n$ respectively.
- Freeness means that Z is free of additive relations and W is free of multiplicative relations.
- Rotundity is an *algebraic property* of V related to Schanuel's conjecture. For example, a rotund variety $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ must have $\dim V \geq n$, and similar inequalities hold for certain projections of V.
- The *strong* EAC conjecture is about existence of generic exponential points in free and rotund varieties.

Quasiminimality

Definition

An uncountable structure is said to be *quasiminimal* if every definable subset (in one variable) is either countable or its complement is countable. Here definable can mean first-order definable, or $\mathfrak{L}_{\omega_1,\omega}$-definable or, more generally, any subset which is invariant under all automorphisms (over a countable set of parameters).

Conjecture (Zilber's quasiminimality conjecture)

$\mathbb{C}_{\exp} := (\mathbb{C}; +, \cdot, \exp)$ *is quasiminimal.*

- The set $2\pi i\,\mathbb{Z}$ is definable as the kernel of exp.
- In fact, $\mathbb{Z}$ is definable as the set

$$\{a \in \mathbb{C} : \forall x(\exp(x) = 1 \to \exp(ax) = 1)\}.$$

- An open question: is $\mathbb{R}$ definable in $\mathbb{C}_{\exp}$?

Zilber's pseudo-exponentiation

- Zilber constructed algebraically closed fields of characteristic 0 equipped with a unary function, called *pseudo-exponentiation*, satisfying some of the basic properties of the complex exp (homomorphism, kernel), the analogues of Schanuel's conjecture and the strong EAC conjecture, and the *Countable Closure Property*.
- These can be axiomatised in $\mathfrak{L}_{\omega_1,\omega}(Q)$, where Q is a quantifier for "there are uncountably many".
- Zilber showed that that theory is categorical in uncountable cardinals. In particular, there is a unique model $\mathbb{B}_{\exp}$ of cardinality $2^{\aleph_0}$.

Conjecture (Zilber)

$\mathbb{B}_{\exp} \cong \mathbb{C}_{\exp}$.

- This conjecture is equivalent to **Schanuel + strong EAC**.
- Kirby proved that if $\mathbb{B}_{\exp} \equiv \mathbb{C}_{\exp}$ then $\mathbb{B}_{\exp} \cong \mathbb{C}_{\exp}$.
- $\mathbb{B}_{\exp}$ is quasiminimal, so the above conjecture implies the quasiminimality conjecture.
- Bays and Kirby proved that EAC implies quasiminimality.

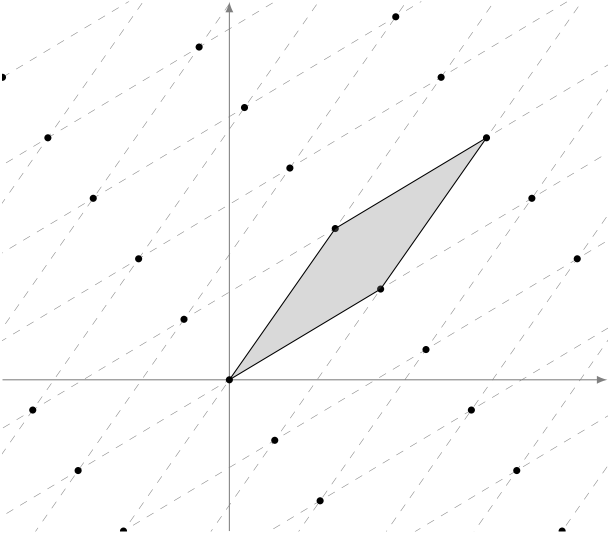
EAC: What is known

- Exponential equations in one variable can be solved. E.g. $e^z = z$ has infinitely many solutions.
- Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$, and let Z and W be the projections of V on $\mathbb{C}^n$ and $(\mathbb{C}^\times)^n$ respectively.
- If $\dim Z = n$ (one says V has a dominant projection to $\mathbb{C}^n$), then $V \cap \Gamma \neq \emptyset$. (Brownawell-Masser, D'Aquino-Fornasiero-Terzo)
- EAC for $\dim Z = 1$. (Mantova-Masser)
- EAC for raising to generic real powers, i.e. Z is a generic real linear space and $V = Z \times W$. (Zilber)

Elliptic curves and abelian varieties

- Let $\Lambda \subseteq \mathbb{C}$ be a lattice of rank 2, e.g. $\mathbb{Z} + i\,\mathbb{Z}$.
- The quotient $\mathbb{C}/\Lambda$ is a torus.
- It can be embedded into the projective plane $\mathbb{P}_2(\mathbb{C})$. The embedding is given by $\exp_E : z \mapsto [1 : \wp(z) : \wp'(z)/2]$ where $\wp$ is the Weierstrass $\wp$-function associated to the lattice Λ.
- An elliptic curve $E \subseteq \mathbb{P}_2$ satisfies a cubic equation (in affine coordinates $y^2 = 4x^3 - g_2 x - g_3$).
- Thus, elliptic curves are connected projective algebraic groups of dimension 1.
- Abelian varieties are higher dimensional analogues of elliptic curves: connected projective algebraic groups.
- When a quotient $\mathbb{C}^g/\Lambda$ (with Λ a lattice of rank $2g$) can be embedded in a projective space, we get an abelian variety.
- The group structure of an abelian variety is commutative.
- Think of products of elliptic curves.

A lattice



EAC for abelian varieties: dominant projection

Let A be an abelian variety of dimension n and let $\exp_A : \mathbb{C}^n \to A$ be its exponential map. Its kernel is a lattice $\Lambda \subseteq \mathbb{C}^n$ of rank $2n$.
There is a Schanuel conjecture and an EAC conjecture in this setting too.

Theorem (A.-Kirby-Mantova)

Let $V \subseteq \mathbb{C}^n \times A$ be an algebraic subvariety with dominant projection to $\mathbb{C}^n$, that is, its projection to $\mathbb{C}^n$ has dimension n. Then there is $\boldsymbol{z} \in \mathbb{C}^n$ such that $(\boldsymbol{z}, \exp_A(\boldsymbol{z})) \in V$.
Moreover, we can locally parametrise all sufficiently large exponential points in V be points of Λ.

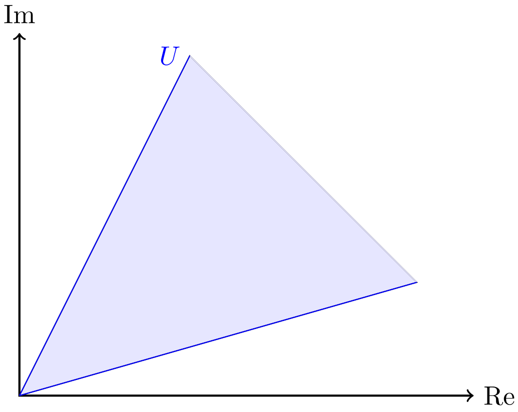
- For example, $\wp'(\wp(z)^2) = z$ has infinitely many solutions.
- Brownawell-Masser (and D'Aquino-Fornasiero-Terzo) used Newton's iterative method to approximate solutions, and in particular Kantorovich's theorem which gives criteria for these approximations to converge to an actual solution.
- Our approach is more geometric. It also works for the usual complex exponentiation. In both cases we also locally describe all sufficiently large exponential points.

Proof

- Clearly, $\dim V \geq n$. We may replace V with a subvariety and assume $\dim V = n$.
- Let $\alpha : U \to A$ be an algebraic map whose graph is contained in V.
- Here $U \subseteq \mathbb{C}^n$ is a neighbourhood of infinity with a branch cut removed. For simplicity, we work with a smaller set U, which is a convex simply connected cone at infinity, i.e. whenever $\boldsymbol{z} \in U$ and $t \geq 1$, we have $t\boldsymbol{z} \in U$.
- α is holomorphic on U.
- Since $(\boldsymbol{z}, \alpha(\boldsymbol{z})) \in V$ it suffices to solve the equation

$$\exp_A(\boldsymbol{z}) = \alpha(\boldsymbol{z}).$$

A cone in $\mathbb{C}$



Mapping solutions to the lattice

- Since U is simply connected, we can choose a holomorphic branch of logarithm of α, which we denote by $G : U \to \mathbb{C}^n$.
- We can pick a fundamental domain D of Λ, and a branch of logarithm $\mathrm{Log}_A : A \to D$, and assume $G = \mathrm{Log} \circ \alpha$.
- In particular, since D is bounded, so is G.
- Define a map $F : U \to \mathbb{C}^n$ by

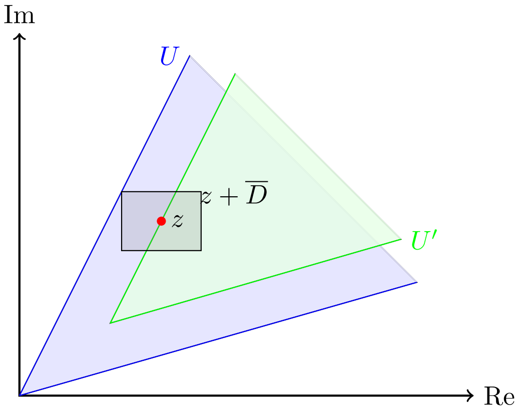
$$F : \boldsymbol{z} \mapsto \boldsymbol{z} - G(\boldsymbol{z}) = \boldsymbol{z} - \mathrm{Log}\,\alpha(\boldsymbol{z}).$$

- Clearly, $\boldsymbol{z} \in U$ solves $\exp_A(\boldsymbol{z}) = \alpha(\boldsymbol{z})$ if and only if $F(\boldsymbol{z}) \in \Lambda$.
- Thus, to prove existence of solutions we need to show that the image $F(U)$ contains lattice points.

The image $F(U)$

- Since G is bounded, by Cauchy estimates, all of its first partial derivatives are bounded. In fact, shrinking U we may assume the first partial derivatives are arbitrarily small, i.e. $\|\mathrm{d}G(\boldsymbol{z})\| < \varepsilon$ on U for a small ε.
- So $\mathrm{d}F(\boldsymbol{z})$ is close to the identity matrix, hence it is non-singular.
- By the inverse function theorem, F is a local homeomorphism, hence an open map.
- So the image $E := F(U)$ is open and connected.
- Moreover, $F(\boldsymbol{z}) - \boldsymbol{z} = -G(\boldsymbol{z})$ is bounded, so F behaves like a translation near points at infinity. So E and U cannot differ by a large set.
- Let $U' := \{\boldsymbol{z} \in U : \overline{z + D} \subseteq U\}$. Then $U' \subseteq E$.
- U' is a cone, hence it contains infinitely many lattice points. Thus, $E \cap \Lambda$ is infinite, and we get infinitely many solutions to $\exp_A(\boldsymbol{z}) = \alpha(\boldsymbol{z})$.

The cones U and U' for $n = 1$



Finding all solutions in U

Lemma

F is injective on U.

Proof.

Assume for some $\boldsymbol{z}^1, \boldsymbol{z}^2 \in U$ we have $F(\boldsymbol{z}^1) = F(\boldsymbol{z}^2)$. Recall that $G(\boldsymbol{z}) = \boldsymbol{z} - F(\boldsymbol{z})$. So we have $\boldsymbol{z}^1 - \boldsymbol{z}^2 = G(\boldsymbol{z}^1) - G(\boldsymbol{z}^2)$. Since the line segment $[\boldsymbol{z}^1, \boldsymbol{z}^2]$ is entirely contained in U, by the mean value theorem

$$|\boldsymbol{z}^1 - \boldsymbol{z}^2| = |G(\boldsymbol{z}^1) - G(\boldsymbol{z}^2)| = O\left(\max_{\boldsymbol{z} \in [\boldsymbol{z}^1, \boldsymbol{z}^2]} \|\mathrm{d}G(\boldsymbol{z})\| \right) \cdot |\boldsymbol{z}^1 - \boldsymbol{z}^2|.$$

Since $\|\mathrm{d}G(\boldsymbol{z})\|$ is small, we must have $\boldsymbol{z}^1 = \boldsymbol{z}^2$. □

Corollary

The map F has a holomorphic inverse $S : E \to U$. All solutions of $\exp_A(\boldsymbol{z}) = \alpha(\boldsymbol{z})$ with $\boldsymbol{z} \in U$ are given by $\boldsymbol{z} = S(\lambda)$ with $\lambda \in E \cap \Lambda$. Asymptotically we have $S(\boldsymbol{x}) = \boldsymbol{x} + \operatorname{Log} \alpha(\boldsymbol{x}) + o(1)$ as $|\boldsymbol{x}| \to \infty$ with $\boldsymbol{x} \in E$.

Finding all large exponential points in V

- In general, we have to work with large domains U – neighbourhoods of points at infinity with a branch cut removed.
- If the projection $\pi : V \to \mathbb{C}^n$ has degree d, then V is covered by the graphs of d branches of α.
- We can find all large solutions for each branch. Altogether, there will be roughly d solutions in each fundamental domain near points at infinity.

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Theorem (A.-Kirby-Mantova)

Let $V \subseteq \mathbb{C}^n \times A$ be an irreducible variety of dimension n with dominant projection $\pi : V \to \mathbb{C}^n$. Let $d := \deg \pi$. We embed $\mathbb{C}^n$ in the projective space $\mathbb{P}_n$ in the usual way.
Then for almost all points at infinity $\boldsymbol{c} \in \mathbb{P}_n \setminus \mathbb{C}^n$, there is an open set $U^* \subseteq \mathbb{P}^n$ containing $\boldsymbol{c}$ such that the points $(\boldsymbol{z}, \exp_A(\boldsymbol{z})) \in V$ with $\boldsymbol{z} \in U := U^* \cap \mathbb{C}^n$ are parametrised by d copies of $\Lambda \cap U$.

The case of algebraic tori

Now consider the exponential map $\exp : \mathbb{C}^n \to (\mathbb{C}^\times)^n$. Its kernel is the lattice $\Lambda = (2\pi i\,\mathbb{Z})^n$, which has rank n.

Theorem

Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be a variety of dimension n with dominant projection $\pi : V \to \mathbb{C}^n$. Let $d := \deg \pi$. We embed $\mathbb{C}^n$ in projective space $\mathbb{P}_n$ in the usual way.
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Proof

- Extract an algebraic map $\alpha : U \to (\mathbb{C}^\times)^n$, where $U \subseteq \mathbb{C}^n$ is chosen such that the the coordinates are roughly proportional to each other. Moreover, we can choose U so that $U \cap \Lambda \neq \emptyset$.
- Then we can understand the growth of α. In particular, $G(\boldsymbol{z}) = \mathrm{Log}(\alpha(\boldsymbol{z}))$ has logarithmic growth.
- As before, define $F : U \to \mathbb{C}^n : \boldsymbol{z} \mapsto \boldsymbol{z} - G(\boldsymbol{z})$.
- Using Cauchy estimates, we can bound the first partial derivatives of G. As in the abelian case, $\|\mathrm{d}G(\boldsymbol{z})\|$ can be taken to be arbitrarily small on U.
- Hence, F is injective and open, and $E := F(U)$ contains a narrower cone $U' \subseteq U$.
- U' also contains infinitely many lattice points, so $\exp(\boldsymbol{z}) = \alpha(\boldsymbol{z})$ has infinitely many solutions.
- The inverse of F parametrises this solutions by lattice points.

The cones U and U' for $n = 1$